Suggested Solution to Exercise 2

1. Find the hyperplanes specified by the directions $\boldsymbol{\xi}$ and points p as shown

   (a) $\boldsymbol{\xi} = (1, 0, 2)$; $\mathbf{p} = (0, 1, -4)$,

   (b) $\boldsymbol{\xi} = (11, 3, 4, 0)$; $\mathbf{p} = (1, -1, -2, -5)$;

   (c) $\boldsymbol{\xi} = (2, -1)$; $\mathbf{p} = (3, 3)$.

   Solution.

   (a) The hyperplane is given by the following equation

   $$(1, 0, 2) \cdot ((x, y, z) - (0, 1, -4)) = 0 \ ,$$

   that is, $x + 2z + 8 = 0$.

   (b) The hyperplane is given by the following equation

   $$(11, 3, 4, 0) \cdot ((x, y, z, w) - (1, -1, -2, -5)) = 0 \ ,$$

   that is, $11x + 3y + 4z = 0$.

   (c) The hyperplane is given by the following equation

   $$(2, -1) \cdot ((x, y) - (3, 3)) = 0 \ ,$$

   that is, $2x - y - 3 = 0$.

2. Find the plane passing through $(0, -1, 1)$ and parallel to the plane $x + y + 4z = 8$.

   Solution. Since the plane is parallel to the given plane, its normal direction is equal to that of the given plane, i.e. $(1, 1, 4)$. Therefore, the plane is given by the following equation:

   $$(1, 1, 4) \cdot ((x, y, z) - (0, -1, -1)) = 0 \ ,$$

   which is simplified to $x + y + 4z - 3 = 0$.

3. Find all planes passing through $(2, -3, 0)$ and intersecting the plane $x - 4y + 2z = -1$ vertically.

   Solution. Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ be a normal direction of any of such planes (and hence $\boldsymbol{\xi} \neq \mathbf{0}$). Then intersecting the plane vertically means $(\xi_1, \xi_2, \xi_3) \cdot (1, -4, 2) = 0$ which yields

   $$\xi_1 = 4\xi_2 - 2\xi_3$$

   Therefore, these planes are described by the following equation

   $$(4\xi_2 - 2\xi_3, \xi_2, \xi_3) \cdot ((x, y, z) - (2, -3, 0)) = 0$$

   i.e.

   $$(4\xi_2 - 2\xi_3)(x - 2) + \xi_2(y + 3) + \xi_3 z = 0 \ , \quad (\xi_2, \xi_3) \neq (0, 0) \ .$$

   Note. Restricting to unit vectors, we can write $(\xi_2, \xi_3) = (\cos\theta, \sin\theta)$ so these planes are described by a one parameter family

   $$(4 - 2\tan\theta)(x - 2) + (y + 3) + \tan\theta z = 0 \ .$$

4. Find the hyperplanes passing through the following points:

   (a) $(1,9),\ (2,-12)$,

   (b) $(1,0,-1),\ (2,0,3),\ (1,1,0)$.

   (c) $(1,1,1),\ (1,-1,1),\ (-1,1,1)$.

   (d) $(1,1,0,-4),\ (0,0,0,3),\ (0,-1,0,-1),\ (4,2,0,0)$.

   Solution.

   (a) Let $\xi = (\xi_1, \xi_2)$ be a normal direction of such plane. The only requirement is

   $$(\xi_1, \xi_2) \cdot ((2,-12) - (1,9)) = 0 \ ,$$

   that is, $\xi_1 = -21\xi_2$. Therefore, we can choose the normal direction to be $(-21, 1)$, and the plane is given by the equation

   $$(-21, 1) \cdot ((x, y) - (1, 9)) = 0 \ ,$$

   that is, $-21x + y + 12 = 0$.

   (b) Let $\xi = (\xi_1, \xi_2, \xi_3)$ be a normal direction of such plane. Then

   $$(\xi_1, \xi_2, \xi_3) \cdot ((2,0,3) - (1,0,-1)) = 0 \ , \quad (\xi_1, \xi_2, \xi_3) \cdot ((1,1,0) - (1,0,-1)) = 0 \ ,$$

   that is $\xi_1 + 4\xi_3$ and $\xi_2 + \xi_3 = 0$, so $(\xi_1, \xi_2, \xi_3) = (-4, -1, 1)\xi_3$. Therefore, a normal direction is given by $(-4, -1, 1)$, and the plane is $(-4,-1,1)\cdot((x,y,z)-(1,0,-1)) = 0$, or, $-4x - y + z + 6 = 0$.

   (c) Let $\xi = (\xi_1, \xi_2, \xi_3)$ be a normal direction of such plane. Then

   $$(\xi_1, \xi_2, \xi_3) \cdot ((1,-1,1) - (1,1,1)) = 0 \ , \quad (\xi_1, \xi_2, \xi_3) \cdot ((-1,1,1) - (1,1,1)) = 0 \ .$$

   It implies $-2\xi_2 = 0$ and $-2\xi_1 = 0$. Thus $(\xi_1, \xi_2, \xi_3) = (0,0,1)\xi_3$. A normal direction is given by $(0,0,1)$, and the plane is $(0,0,1) \cdot ((x,y,z) - (1,1,1)) = 0$, or $z - 1 = 0$.

   (d) Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be a normal direction of such plane. Then

   $$(\xi_1, \xi_2, \xi_3, \xi_4) \cdot ((1,1,0,-4) - (0,0,0,3)) = 0 \ ,$$
   $$(\xi_1, \xi_2, \xi_3, \xi_4) \cdot ((0,-1,0,-1) - (0,0,0,3)) = 0$$

   and

   $$(\xi_1, \xi_2, \xi_3, \xi_4) \cdot ((4,2,0,0) - (0,0,0,3)) = 0 \ ,$$

   give $\xi_1+\xi_2-7\xi_4 = 0,\ -\xi_2-4\xi_4 = 0,\ 4\xi_1+2\xi_2-3\xi_4 = 0$. It follows that $(\xi_1, \xi_2, \xi_3, \xi_4) = (0,0,1,0)\xi_3$. Therefore, a normal direction is given by $(0,0,1,0)$, and the plane is $(0,0,1,0) \cdot ((x,y,z,w) - (0,0,0,3)) = 0$, or, $z = 0$

5. Show that the plane passing through $\mathbf{a}, \mathbf{b}$ and the origin is given by the determinant equation

   $$\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \ .$$

   What happens if $\mathbf{a}$ and $\mathbf{b}$ are collinear?

Solution.

Expanding

$$\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \ .$$

gives the form $Ax + By + Cz = 0$.

When $(A, B, C) \neq (0, 0, 0)$, then this defines a plane passing through origin.

Moreover, since

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \ ,$$

the plane also passes through $\mathbf{a}$.

Similarly,

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \ ,$$

implies that the plane also passes through b. Therefore, this determinant equation defines a plane passes through $\mathbf{a}, \mathbf{b}$ and origin.

If $\mathbf{a}, \mathbf{b}$ are collinear, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, and hence $(A, B, C) = (0, 0, 0)$.

6. Find the distance from $\mathbf{P}$ to the hyperplane H and the point $\mathbf{Q}$ on H realizing the distance:

   (a) $(1, 2)$, $\quad x + 6y = 0$.

   (b) $(1, 2, 4)$, $\quad x - 2y = -1$,

   (c) $(2, 4, 6, 8)$, $\quad x - 3y - 4z + w = -6$.

   Solution.

   (a) Let $\mathbf{Q} = (q_1, q_2)$. Since $\mathbf{Q}$ realizes the distance, $((1, 2) - (q_1, q_2)) \perp \{x + 6y = 0\}$ implies $(1 - q_1, 2 - q_2) = \lambda(1, 6)$ for some λ. Hence $1 - q_1 = \lambda$ and $2 - q_2 = 6\lambda$. Also, since $\mathbf{Q}$ lies on H, $q_1 + 6q_2 = 0$. Solving these three equations, we get $q_1 = -4/37$ and $q_2 = 6q_1 - 4 = -292/37$.
   So $\mathbf{Q} = (-4/37, -292/37)$, and

   $$d = |(1, 2) - q| = \sqrt{\left(\frac{41}{37}\right)^2 + \left(\frac{366}{37}\right)^2} = \frac{\sqrt{135637}}{37}.$$

   (b) Let $\mathbf{Q} = (q_1, q_2, q_3)$. We have

   $$(1 - q_1, 2 - q_2, 4 - q_3) = \lambda(1, -2, 0) \ ,$$

   for some λ. Thus $1 - q_1 = \lambda$, $2 - q_2 = -2\lambda$, $4 - q_3 = 0$. Also, since $\mathbf{Q}$ lies on H,$q_1 - 2q_2 = -1$. Solving to get

   $$\mathbf{Q} = \left(-\frac{7}{5}, \frac{34}{5}, 4\right) \ .$$

   So

   $$d = |(1, 2, 4) - q| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(-\frac{24}{5}\right)^2 + 0^2} = \frac{12\sqrt{5}}{5} \ .$$

(c) Let $\mathbf{Q} = (q_1, q_2, q_3, q_4)$. We have

$$(2 - q_1, 4 - q_2, 6 - q_3, 8 - q_4) = \lambda(1, -3, -4, 1)$$

for some λ. Also, since $\mathbf{Q}$ lies on H, $q_1 - 3q_2 - 4q_3 + q_4 = -1$. Solving this linear system of five equations for five variables to get

$$q_1 = \frac{79}{27},\ q_2 = \frac{11}{9},\ q_3 = \frac{62}{27},\ q_4 = \frac{241}{27}\ .$$

So $\mathbf{Q} = \Big(\frac{79}{27}, \frac{11}{9}, \frac{62}{27}, \frac{241}{27}\Big)$, and

$$d = |(2,4,6,8) - q| = \sqrt{\Big(-\frac{25}{27}\Big)^2 + \Big(\frac{25}{9}\Big)^2 + \Big(\frac{100}{27}\Big)^2 + \Big(-\frac{25}{27}\Big)^2} = \frac{25\sqrt{3}}{9}\ .$$

7. Find two parametric forms of the straight lines described by the following system using x and z as the parameters respectively:

$$\begin{cases} 2x - 6y = 0, \\ 3x + 5y - 6z = -1\ . \end{cases}$$

Solution. (i) Using x as the parameter. Then $2x - 6y = 0$ implies $y = \frac{x}{3}$, and $3x + 5y - 6z = -1$ becomes $3x + \frac{5x}{3} - 6z = -1$. Hence $z = \frac{14x+3}{18}$.
The parametric form of the straight line is given by

$$(x, y, z) = \Big(x, \frac{x}{3}, \frac{14x+3}{18}\Big) = \Big(1, \frac{1}{3}, \frac{7}{9}\Big)x + \Big(0, 0, \frac{1}{6}\Big)\ ,$$

where $x \in \mathbb{R}$ is the parameter.

(ii) Using z as the parameter. Then write $3x + 5y - 6z = -1$ $3x + 5y = -1 + 6z$. We have the following system of linear equations in x, y: $x - 3y = 0,\ 3x + 5y = -1 + 6z$, which gives $-14y = 1 - 6z$. Thus $y = \frac{6z-1}{14}$. Therefore, $x = 3y = \frac{18z-3}{14}$. The parametric form of the straight line is given by

$$(x, y, z) = \Big(\frac{18z-3}{14}, \frac{6z-1}{14}, z\Big) = \Big(\frac{9}{7}, \frac{3}{7}, 1\Big)z + \Big(-\frac{3}{14}, -\frac{1}{14}, 0\Big)\ ,$$

where $z \in \mathbb{R}$ is the parameter.

8. Find the point of intersection (if any) of the lines and planes:

(a) $(0, 2) + t(1, 1)\ ,\quad x + 3y = 1\ .$

(b) $(7, 3, 0) + t(-4, 6, 5)\ ,\quad 4x + y + 2z = 17\ .$

(c) $(15, 10, 5) + t(7, 12, -4)\ ,\quad x + y + z = 45.$

Solution. (a) Plug $x = t, y = 2 + t$ into the equation to get $t + 6 + 3t = 1$ which yields $t = -5/4$. So the intersection point is $(0, 2) + \frac{-5}{4}(1, 1) = (-5/4, 3/4)$.

(b) Using $x = 7 - 4t, y = 3 + 6t, z = 5t$ to get $31 = 17$ from the equation. There is no solution. We conclude that the line does not intersect the plane.

(c) Solve to get $t = 1$ so the point of intersection is $(22, 22, 1)$.

9. The angle between two planes is defined to be the angle between their normal directions (lying in $[0, \pi)$). Find these angles in the following cases:

   (a) $2x - y + z = 13$ and $x + y - z = 1$.

   (b) $x + 11y - 5z = 0$ and the xy-plane.

   Solution.

   (a) The normals on the corresponding planes are given by $(2, -1, 1)$ and $(1, 1, -1)$ respectively.

   Therefore,
$$\cos\theta = \frac{(2, -1, 1) \cdot (1, 1, -1)}{|(2, -1, 1)||(1, 1, -1)|} = 0$$
   Hence, the angle is $\pi/2$, that is, these two planes are perpendicular.

   (b) The normals on the corresponding planes are given by $(1, 11, -5)$ and $(0, 0, 1)$ respectively.

   Therefore,
$$\cos\theta = \frac{(1, 11, -5) \cdot (0, 0, 1)}{|(1, 11, -5)||(0, 0, 1)|} = -\frac{5}{\sqrt{147}}$$
   Hence, the angle is $\cos^{-1}\left(-\frac{5}{\sqrt{147}}\right) \in (\pi/2, \pi)$.

10. Find the straight lines passing through $(1, -2, 3)$ and hitting the plane $x - y + 6z = -1$ at right angle. Find the point of intersection too.

    Solution. A normal direction of the given plane is $(1, -1, 6)$. Therefore, the required straight line is parallel to the normal direction, i.e. $(1, -1, 6)$, and passes through $(1, -2, 3)$. It takes the parametric form
$$(x, y, z) - (1, -2, 3) = t(1, -1, 6)\ , \quad t \in \mathbb{R}\ .$$
    Since the line hits the plane, let $(x_0, y_0, z_0) = (1, -2, 3) + t_0(1, -1, 6)$ be the intersection point. Then
$$(1 + t_0) - (-2 - t_0) + 6(3 + 6t_0) = -1$$
    which implies $38t_0 = -22$, and hence $t_0 = -\frac{11}{19}$.

    Therefore, the intersection point is
$$(1, -2, 3) - \frac{11}{19}(1, -1, 6) = \frac{1}{19}(8, -27, -9)\ .$$

11. Find the distance between the two straight lines in the following cases:

    (a) $(1, 2, 3) + t(0, 1, 0)$ and $s(1, 0, -2)$.

    (b) $(1, 0, 1, 0) + t(-1, 0, 0, 0)$ and $(-1, -1, 0, 1) + s(0, 2, -1, 0)$.

Hint: The distance is realized by the line segment that is perpendicular to both lines.

Solution. (a) Let $(1, 2+t, 3)$ and $(s, 0, -2s)$ be respectively the points on the first and the second line realizing the distance. It is characterized by

$$(1-s, 2+t, 3+2s)\cdot(0,1,0) = 0\ , \quad (1-s, 2+t, 3+2s)\cdot(1,0,-2) = 0\ .$$

Solving it we get $t = -2$ and $s = -1$. So the points are given by $(1, 0, 3)$ and $(-1, 0, 2)$ and the distance is $\sqrt{5}$.

(b) The points are $(1-t, 0, 1, 0)$ and $(-1, 2s-1, -s, 1)$ respectively. The perpendicular conditions are

$$(2-t, -2s+1, 1+s, -1)\cdot(-1,0,0,0) = 0\ , \quad (2-t, -2s+1, 1+s, -1)\cdot(0,2,-1,0) = 0\ .$$

Solve to get $t = 2$ and $s = 3/5$ and the points are $(-1, 0, 1, 0)$ and $(-1, 1/5, -3/5, 1)$. The distance is $3\sqrt{10}/5$.

12. Propose a definition of the projection of $P(x, y, z)$ on a straight line passing through the origin and then find a formula for it.

Solution. The projection of $P(x_0, y_0, z_0)$ on a straight line L described by $t(a, b, c),\ t \in \mathbb{R}$, is the point $Q(x, y, z)$ satisfying (i) Q lies on L and (ii) $P - Q = (x_0 - x, y_0 - y, z_0 - z)$ is perpendicular to (a, b, c).

By (i), $Q = t_0(a, b, c)$ for some $t_0 \in \mathbb{R}$. By (ii), $(x_0 - t_0 a, y_0 - t_0 b, z_0 - t_0 c)\cdot(a, b, c) = 0$, which implies $(x_0, y_0, z_0)\cdot(a, b, c) = t_0|(a, b, c)|^2$, and hence $t_0 = \dfrac{(x_0, y_0, z_0)\cdot(a, b, c)}{|(a, b, c)|^2}$. The projection Q is given by

$$Q = \frac{(x_0, y_0, z_0)\cdot(a, b, c)}{|(a, b, c)|^2}(a, b, c)\ .$$

13. Find the three medians of the triangle $A(0, 0),\ B(2, 6),\ C(4, -4)$ and verify that they meet at a point.

Solution.

We first find the midpoints of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ respectively. Midpoint of $\overline{BC} = \frac{1}{2}((2, 6) + (4, -4)) = (3, 1)$, midpoint of $\overline{CA} = \frac{1}{2}((4, -4) + (0, 0)) = (2, -2)$, and midpoint of $\overline{AB} = \frac{1}{2}((0, 0) + (2, 6)) = (1, 3)$. We then find the equations of three medians: The median l_1 from A to $\overline{BC}$ is given by $t(3, 1),\ t \in [0, 1]$, the median l_2 from B to $\overline{CA}$ is given by $(2, 6) + s((2, -2) - (2, 6)) = (2, 6) + s(0, -8),\ s \in [0, 1]$ and the median l_3 from C to $\overline{AB}$ is given by $(4, -4) + z((1, 3) - (4, -4)) = (4, -4) + z(-3, 7),\ z \in [0, 1]$.

Suppose l_1 and l_2 meet at $P = (x, y)$. Write $(x, y) = t_0(3, 1)$ for some $t_0 \in [0, 1]$ as $P \in l_1$, then since $P \in l_2$ also, there exists $s_0 \in [0, 1]$ such that $t_0(3, 1) = (2, 6) + s_0(0, -8)$, i.e.

$$\begin{cases} 3t_0 = 2 \\ t_0 = 6 - 8s_0 \end{cases}$$

which is solved to get $t_0 = \dfrac{2}{3}$ and $s_0 = \dfrac{2}{3}$. Therefore, $P = \dfrac{2}{3}\Big(3, 1\Big) = \Big(2, \dfrac{2}{3}\Big)$.

To show l_3, l_2, l_1 meet at a point, it suffices to show that $P \in l_3$: in other words, whether there exists $z_0 \in [0,1]$ such that $\left(2, \dfrac{2}{3}\right) = (4,-4) + z_0(-3,7)$, i.e.

$$\begin{cases} 2 = 4 - 3z_0 \\ \dfrac{2}{3} = -4 + 7z_0 \end{cases}$$

Both equations have solution $z_0 = \dfrac{2}{3}$. Therefore, $P \in l_3$, and hence l_3, l_2, l_1 meet at a point.

14. Let $A(1,0)$, $B(2,3)$, $C(4,4)$ be a triangle. Determine its altitude from A to BC and from B to AC.

Solution.

The altitude from A to $\overline{BC}$ is perpendicular to $(4,4)-(2,3)=(2,1)$. We may take its direction to be $(-1,2)$. The line passing through A with direction $(2,1)$ has the parametric form $(1,0)+t(-1,2),\ t \in \mathbb{R}$. Meanwhile, the line passes through B, C is given by $(2,3)+s((4,4)-(2,3)) = (2,3)+s(2,1),\ s \in \mathbb{R}$. Suppose these two lines meet at $D=(x,y)$. There exist $t_0, s_0 \in \mathbb{R}$ such that $(x,y) = (1,0)+t_0(-1,2) = (2,3)+s_0(2,1)$, i.e.

$$\begin{cases} 1 - t_0 = 2 + 2s_0 \\ 2t_0 = 3 + s_0 \end{cases}$$

which has the solution $t_0 = 1$ and $s_0 = -1$. Therefore, $P = (1,0)+(-1,2) = (0,2)$, and hence the altitude $\overline{AD}$ is given by $(1,0)+t(-1,2),\ t \in [0,1]$

The altitude from B to $\overline{AC}$ is perpendicular to $(4,4)-(1,0) = (3,4)$. We may take its direction to be $(-4,3)$. The line passing through B with direction $(-4,3)$ has the parametric form $(2,3)+t(-4,3),\ t \in \mathbb{R}$. Meanwhile, the line passes through A, C is given by $(1,0)+s((4,4)-(1,0)) = (1,0)+s(3,4),\ s \in \mathbb{R}$. Suppose these two lines meet at $E=(x,y)$. Then there exist $t_0, s_0 \in \mathbb{R}$ such that $(x,y) = (2,3)+t_0(-4,3) = (1,0)+s_0(3,4)$, i.e.

$$\begin{cases} 2 - 4t_0 = 1 + 3s_0 \\ 3 + 3t_0 = 4s_0 \end{cases}$$

which has the solution $t_0 = -\dfrac{1}{5}$ and $s_0 = \dfrac{3}{5}$. Therefore, $E = (2,3) - \dfrac{1}{5}(-4,3) = \left(\dfrac{14}{5}, \dfrac{12}{5}\right)$, and hence the altitude $\overline{BE}$ is given by $(2,3)+t\left(\dfrac{4}{5}, -\dfrac{3}{5}\right),\ t \in [0,1]$.

15. * Let $A(3,4)$, $B(0,0)$, $C(2,0)$ be a triangle. Determine its angle bisector from A and from C.

Solution.

Let $\xi = (\xi_1, \xi_2)$ be a direction of the angle bisector at A. Since it bisects $\angle BAC$,

$$\frac{(3,4)\cdot\xi}{5|\xi|} = \frac{((3,4)-(2,0))\cdot\xi}{\sqrt{17}|\xi|} = \frac{(1,4)\cdot\xi}{\sqrt{17}|\xi|}\ ,$$

which implies

$$\sqrt{17}(3\xi_1 + 4\xi_2) = 5(\xi_1 + 4\xi_2)\ .$$

Set $\xi = (16, 13-\sqrt{17})$. The line passing through A and parallel to the angle bisector of $\angle BAC$ has the form $(3,4)+t(16, 13-\sqrt{17}),\ t \in \mathbb{R}$. Meanwhile, the line passes through B, C

is given by $(0,0)+s((2,0)-(0,0)) = s(2,0),\ s\in\mathbb{R}$. Let $E(x,y)$ be the intersection point of these two lines. There exist $t_0, s_0 \in \mathbb{R}$ such that $(x,y) = (3,4)+t_0(16, 13-\sqrt{17}) = s_0(2,0)$, i.e.

$$\begin{cases} 3+16t_0 = 2s_0 \\ 4+(13-\sqrt{17})t_0 = 0 \end{cases}$$

which has the solution $t_0 = -\dfrac{4}{13-\sqrt{17}} = -\dfrac{(13+\sqrt{17})}{38}$ and $s_0 = -\dfrac{47+8\sqrt{17}}{38}$. Therefore, $E = (x,y) = (3,4) - \dfrac{(13+\sqrt{17})}{38}(16, 13-\sqrt{17})$, and hence the angle bisector $\overline{AE}$ is given by $(3,4)+t\Big(-\dfrac{64}{13-\sqrt{17}}, -1\Big),\ t\in[0,1]$.

Let $\phi = (\phi_1, \phi_2)$ be a direction of the angle bisector at C. Since it bisects $\angle BCA$,

$$\frac{(2,0)\cdot\phi}{2|\phi|} = \frac{((3,4)-(2,0))\cdot\phi}{\sqrt{17}|\phi|} = \frac{(1,4)\cdot\phi}{\sqrt{17}|\phi|} \ ,$$

which implies $\sqrt{17}(\phi_1) = (\phi_1 + 4\phi_2)$.

Set $\phi = (4, \sqrt{17}-1)$. The line passing through A and parallel to the angle bisector of $\angle BAC$ has the form $(2,0)+t(4,\sqrt{17}-1),\ t\in\mathbb{R}$. Meanwhile, the line passes through A, B is given by $(0,0)+s((3,4)-(0,0)) = s(3,4),\ s\in\mathbb{R}$

Let $F(x,y)$ be the intersection point of these two lines. Then there exist $t_0, s_0\in\mathbb{R}$ such that $(x,y) = (2,0)+t_0(4,\sqrt{17}-1) = s_0(3,4)$, i.e.

$$\begin{cases} 2+4t_0 = 3s_0 \\ (\sqrt{17}-1)t_0 = 4s_0 \end{cases}$$

which has the solution $t_0 = -\dfrac{8}{3\sqrt{17}-19}$ and $s_0 = -\dfrac{2(\sqrt{17}-1)}{3\sqrt{17}-19}$. Therefore, $F = (x,y) = (2,0) - \dfrac{8}{3\sqrt{17}-19}(4, \sqrt{17}-1)$, and hence the angle bisector $\overline{CF}$ is given by

$$(2,0) - t\Big(-\frac{32}{3\sqrt{17}-19}, -\frac{8(\sqrt{17}-1)}{3\sqrt{17}-19}\Big),\ t\in[0,1]\ .$$

16. Find the "standard forms" of the following quadratic equations and describe their solution sets.

(a) $x^2 - 2xy + 2y = 0$,

(b) $x^2 + 2xy + y^2 + 2y = -6$,

(c) * $5x^2 + 4y^2 - 2xy + ax = -1$, $\quad a\in\mathbb{R}$.

Solution.

(a) The matrix of this quadratic form is

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

whose determinant is equal to -1. This curve is a hyperbola.
To find its standard form, we note that its characteristic equation is $(1-\lambda)(-\lambda)-1=0$. Solve it to get two eigenvalues $\lambda_1 = \dfrac{1-\sqrt{5}}{2}$, $\lambda_2 = \dfrac{1+\sqrt{5}}{2}$.
For each of the above eigenvalues, we solve for an associated eigenvector. First, from

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Big(\frac{1-\sqrt{5}}{2}\Big) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ,$$

we get $(u_1, u_2) = \Big(\dfrac{-1+\sqrt{5}}{2}, 1\Big)$. Similarly, an eigenvector for λ_2 is found to be
For λ_2, let $\mathbf{v} = (v_1, v_2)^t$ be an eigenvector, then it satisfies $\Big(\dfrac{-1-\sqrt{5}}{2}, 1\Big)$.

Therefore, the following change of variables will remove the mixed term xy:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} .$$

Explicit computations by substitution yields

$$\begin{aligned} x^2 - 2xy + 2y &= \Big(\frac{-1+\sqrt{5}}{2}x' - \frac{1+\sqrt{5}}{2}y'\Big)^2 - 2\Big(\frac{-1+\sqrt{5}}{2}x' - \frac{1+\sqrt{5}}{2}y'\Big)(x'+y') + 2(x'+y') \\ &= \frac{3-\sqrt{5}}{2}x'^2 + \frac{3+\sqrt{5}}{2}y'^2 - 2x'y' + 2\frac{-1+\sqrt{5}}{2}x'^2 + 2\frac{1+\sqrt{5}}{2}y'^2 + 2x'y' + 2(x'+y') \\ &= \frac{5-3\sqrt{5}}{2}x'^2 + \frac{5+3\sqrt{5}}{2}y'^2 + 2x' + 2y' \\ &= -a^2x'^2 + b^2y'^2 + 2x' + 2y' \end{aligned}$$

where $a^2 = \dfrac{3\sqrt{5}-5}{2} > 0$ and $b^2 = \dfrac{5+3\sqrt{5}}{2} > 0$.

Finally, let $\tilde{x} = ax'$ and $\tilde{y} = by'$, we have

$$\begin{aligned} x^2 - 2xy + 2y &= -a^2x'^2 + b^2y'^2 + 2x' + 2y' \\ &= -\tilde{x}^2 + \tilde{y}^2 + \frac{2}{a}\tilde{x} + \frac{2}{b}\tilde{y} \\ &= -\left(\tilde{x} - \frac{1}{a}\right)^2 + \left(\tilde{y} + \frac{1}{b}\right)^2 - \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \\ &= -u^2 + v^2 + c , \end{aligned}$$

where $u = \tilde{x} - a^{-1}$, $v = \tilde{y} + b^{-1}$ and $c = -(a^{-2} + b^{-2})$.

(b) The matrix associated to the quadratic form is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .$$

Its determinant vanishes, so the curve is a parabola. To remove the mixed term, we first note that its eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = 2$.

For each of the above eigenvalues, we solve for an associated eigenvector:

For λ_1, an eigenvector is given by $(1, -1)$. For λ_2, an eigenvector is given by $(1, 1)$. Therefore, by the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} ,$$

we have

$$\begin{aligned} x^2 + 2xy + y^2 + 2y + 6 &= (x' + y')^2 + 2(x' + y')(-x' + y') + (-x' + y')^2 + 2(-x' + y') + 6 \\ &= (x'^2 + y'^2 + 2x'y') + (-2x'^2 + 2y'^2) + (x'^2 + y'^2 - 2x'y') \\ &\quad +2(-x' + y') + 6 \\ &= 4y'^2 - 2x' + 2y' + 6 \ . \end{aligned}$$

Finally, let $\tilde{x} = -2x'$ and $\tilde{y} = 2y'$, we have

$$\begin{aligned} x^2 + 2xy + y^2 + 2y &= 4y'^2 - 2x' + 2y' + 6 \\ &= \tilde{y}^2 + \tilde{x} + \tilde{y} + 6 \\ &= \left(\tilde{y} + \frac{1}{2}\right)^2 + \tilde{x} + \left(6 - \frac{1}{4}\right) \\ &= u + v^2 + c \ , \end{aligned}$$

where $u = \tilde{x}$, $v = \tilde{y} + 1/2$ and $c = 23/4$.

(c) The matrix of the quadratic form is

$$\begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix} .$$

Its determinant is 19, so the curve is an ellipse.

The two eigenvalues are $\lambda_1 = \dfrac{9 - \sqrt{5}}{2}$ and $\lambda_2 = \dfrac{9 + \sqrt{5}}{2}$. And the first eigenvector is $(\sqrt{5} - 1, 2)$ and the second $(\sqrt{5} + 1, -2))$. By the change of variables:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{5} - 1 & \sqrt{5} + 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

which will cancel the cross term xy. Thus

$$\begin{aligned} &5x^2 + 4y^2 - 2xy + ax + 1 \\ &= 5((\sqrt{5} - 1)x' + (\sqrt{5} + 1)y')^2 + (2x' - 2y')^2 - 2((\sqrt{5} - 1)x' + \\ &(\sqrt{5} + 1)y')(2x' - 2y') + a((\sqrt{5} - 1)x' + (\sqrt{5} + 1)y') + 1 \\ &= (5((\sqrt{5} - 1))^2 x'^2 + 5(\sqrt{5} + 1)^2 y'^2 + 10(\sqrt{5} - 1)(\sqrt{5} + 1)x'y') \\ &+ (4x'^2 + 4y'^2 - 8x'y') - (4(\sqrt{5} - 1)x'^2 - 4((\sqrt{5} + 1))y'^2 \\ &+ 2(-2(\sqrt{5} - 1) + 2(\sqrt{5} + 1))x'y') + a((\sqrt{5} - 1)x' + (\sqrt{5} + 1)y') + 1 \\ &= (5(\sqrt{5} - 1)^2 - 4(\sqrt{5} - 1) + 4)x'^2 + (5(\sqrt{5} + 1)^2 + 4(\sqrt{5} + 1) + 4)y'^2 \\ &+ a((\sqrt{5} - 1)x' + (\sqrt{5} + 1)y') + 1 \\ &= c^2 x'^2 + d^2 y'^2 + a((\sqrt{5} - 1)x' + (\sqrt{5} + 1)y') + 1 \end{aligned}$$

where $c^2 = 5(\sqrt{5}-1)^2 - 4(\sqrt{5}-1) + 4 > 0$ and $d^2 5(\sqrt{5}+1)^2 + 4(\sqrt{5}+1) > 0$.

Finally, let $\tilde{x} = cx'$ and $\tilde{y} = dy'$, we have

$$\begin{aligned}
5x^2 + 4y^2 - 2xy + ax + 1 &= c^2x'^2 + d^2y'^2 + a((\sqrt{5}-1)x' + (\sqrt{5}+1)y') + 1 \\
&= \tilde{x}^2 + \tilde{y}^2 + \frac{a(\sqrt{5}-1)}{c}\tilde{x} + \frac{a(\sqrt{5}+1)}{d}y' + 1 \\
&= u^2 + v^2 + b
\end{aligned}$$

where $u = \tilde{x} + \dfrac{a(\sqrt{5}-1)}{2c}$, $v = \tilde{y} + \dfrac{a(\sqrt{5}+1)}{2d}$ and

$$b = 1 - \left(\frac{a(\sqrt{5}-1)}{2c}\right)^2 - \left(\frac{a(\sqrt{5}+1)}{2d}\right)^2 .$$